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ON THE HITCHCOCK DISTRIBUTION PROBLEM

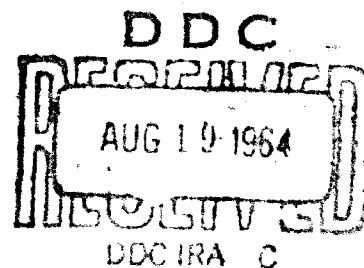
Merrill M. Flood

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## ON THE HITCHCOCK DISTRIBUTION PROBLEM

Merrill M. Flood

### 1. Introduction

Frank L. Hitchcock [1]<sup>1</sup> has offered a mathematical formulation of the problem of determining the most economical manner of distribution of a product from several sources of supply to numerous localities of use. He also suggests a computational procedure for obtaining a solution of his system in any particular case. L. Kantorovitch [2], Tjalling C. Koopmans [3], George B. Dantzig [4b], C. B. Tompkins [5], Julia Robinson [6], Alex Orden [7] and others [4] have also discussed the computational aspects of this problem.

The present paper is concerned only with the mathematical justification of computational procedure, and is limited to one specific method of solution of general validity. No attempt is made to compare the various methods already proposed, either as to their mathematical similarity or as to their relative efficiency in any particular case.<sup>2</sup>

## 2. The Problem

The problem is to find a set of values for the  $mn$  variables  $x_{ij}$ , subject to the following conditions:

$$(2.1) \quad \sum_{i=1}^m x_{ij} = c_j, \quad \sum_{j=1}^n x_{ij} = r_i,$$

$$(2.2) \quad x_{ij} \geq 0,$$

$$(2.3) \quad \sum_{i,j} x_{ij} d_{ij} = \text{minimum}.$$

The numbers  $m$ ,  $n$ ,  $r_i$ ,  $c_j$ , and  $d_{ij}$  are given positive integers<sup>o</sup> with  $\sum c_j = \sum r_i$ . The indices  $i$  and  $j$  are understood always to range over these same integers. It is also assumed, for convenience, that  $m \geq n$ . Any set of values  $x_{ij}$  that satisfies all these conditions is called a solution of the problem.

It will sometimes be more convenient to use an alternative statement of the problem, in matrix notation, as follows:

$$(2.4) \quad M'y \geq b,$$

$$(2.5) \quad y \geq 0,$$

$$(2.6) \quad a'y = \text{minimum}.$$

It is easily seen that the two formulations are equivalent if  $y$ ,  $a$ ,  $b$ , and  $M'$  are defined as follows:

$$y_{n(i-1)+j} = x_{ij},$$

$$a_{n(i-1)+j} = d_{ij},$$

$$M' = \begin{bmatrix} I_n & I_n & \cdots & I_n \\ -I_n & -I_n & \cdots & -I_n \\ J_1 & J_2 & \cdots & J_m \\ -J_1 & -J_2 & \cdots & -J_m \end{bmatrix} \quad b = \begin{bmatrix} c \\ -c \\ r \\ -r \end{bmatrix}$$

where  $I_n$  is the identity matrix of order  $n$ , and  $J_i$  is the  $m \times n$  matrix with all elements zero except for the  $i^{\text{th}}$  row in which each element is unity. Of course,  $y$ ,  $a$ ,  $c$ , and  $r$  are column matrices (or vectors) with components  $y_{n(i-1)+j}$ ,  $a_{n(i-1)+j}$ ,  $c_j$  and  $r_j$ , respectively, and a prime denotes the transpose of a matrix (or vector).

### 3. Fundamental Theorems

There are several fundamental theorems concerning systems of linear inequalities that are useful for this paper. I reproduce their statements here in a form due to A. W. Tucker.<sup>4</sup> The interested reader can find proofs of these theorems, and of others of similar type, in a paper by Gale, Kuhn, and Tucker [4c].

Fundamental Problems: (Here lower case roman letters denote one column vectors, while capitals denote rectangular matrices;  $M$ ,  $a$ , and  $b$  are given but  $d$  is to be determined.)

Problem I. To satisfy the constraints  $Mx \leq a$ ,  $x \geq 0$ , and make  $b'x = d$  for  $d$  maximal in the sense that no  $x$  satisfying the constraints makes  $b'x > d$ .

Problem II. To satisfy the constraints  $M'y \geq b$ ,  $y \geq 0$ , and make  $a'y = d$  for  $d$  minimal in the sense that no  $y$  satisfying the constraints makes  $a'y < d$ .

Problems I and II are said to be dual.

Fundamental Feasibility Theorem:

The constraints in a problem are feasible (i.e., satisfied by some  $x$  or  $y$ ) if and only if the dual problem in homogeneous form (i.e., with  $b=0$  or  $a=0$ ) has a null solution.

Fundamental Existence Theorem:

i.  $x$  and  $y$  are solutions of Problems I and II if and only if they satisfy their constraints in the two problems and make  $a'y = b'x$ . Such  $x$  and  $y$  exist if the constraints in both problems are feasible.

ii. A problem has a solution if and only if its constraints are feasible and its homogeneous form has a null solution.

Fundamental Duality Theorem:

A problem has a solution (for a unique  $d$ ) if and only if the dual problem has a solution (for the same  $d$ ).

4. The Dual and Combined Problems

We note that the problem, as stated in relations (2.4)-(2.6), is a fundamental problem of form II. The dual problem is;

$$(4.1) \quad Mx \leq a,$$

$$(4.2) \quad x \geq 0,$$

$$(4.3) \quad b'x = \text{maximum}.$$

This can be rewritten in a more convenient form, for our present purposes, as follows:<sup>5</sup>

$$(4.4) \quad v_j - u_1 \leq d_{1j},$$

$$(4.5) \quad \sum c_j v_j - \sum r_1 u_1 = \text{maximum},$$

where  $v_j = x_j - x_{n+j}$ , and  $u_1 = -x_{sn+1} + x_{sn+m+1}$ .

Theorem 1. The problem has a solution.

Proof: By the Fundamental Existence Theorem, there is a solution if and only if the constraints are feasible and  $y = 0$  is a solution of the problem when  $b = 0$ . Now  $\sum_j r_1 x_{1j} = \sum_j r_1 \cdot 0 = 0$ , so  $x_{1j} = r_1 c_j / \sum_j r_1 c_j$  satisfies the constraints. When  $b = 0$ , obviously the only values that satisfy the constraints are  $x_{1j} = 0$ , and so the theorem is proved.

By the Fundamental Duality Theorem, we see

Corollary 1A. The dual problem has a solution.

Theorem 2. The numbers  $x_{1j}$ , and  $u_1, v_j$ , are solutions of the problem and the dual, respectively, if and only if they satisfy:

$$(4.6) \quad \sum_j r_1 x_{1j} = r_1, \quad \sum_j c_j x_{1j} = c_j, \quad x_{1j} \geq 0,$$

$$(4.7) \quad d_{1j} + u_1 - v_j \geq 0,$$

$$(4.8) \quad x_{1j}(d_{1j} + u_1 - v_j) = 0.$$

Proof: Since (4.6) and (4.7) are simply the constraints for the problem and the dual, respectively, it remains only to show that (4.8) is equivalent to the condition  $a'y - b'x = 0$ . Now

$$\begin{aligned} a'y - b'x &= \sum_{1,j} x_{1j} d_{1j} - \sum_j c_j v_j + \sum_1 r_1 u_1 \\ &= \sum_{1,j} x_{1j} d_{1j} - \sum_{1,j} x_{1j} v_j + \sum_{1,j} x_{1j} u_1 \\ &= \sum_{1,j} x_{1j} (d_{1j} + u_1 - v_j). \end{aligned}$$

Since each term in this sum is non-negative,  $a'y - b'x = 0$  if and only if  $x_{1j}(d_{1j} + u_1 - v_j) = 0$ .

We refer to the problem of finding values for  $x_{ij}$ ,  $u_i$ , and  $v_j$  that satisfy (4.5)-(4.6) as the "combined problem", and note that the combined problem always has a solution.

## 5. Linear Graphs

It will be convenient, for some purposes, to associate linear graphs [6] with certain subsets of the elements of a matrix  $S = \|s_{hk}\|$ . If  $I$  is a given subset of the elements of  $S$ , we define the  $I$ -graph  $L$  of  $S$  as follows: the vertices of  $L$  are all the points  $(h,k)$  in the Cartesian plane for which  $s_{hk} \in I$ ; the arcs of  $L$  are all line segments joining pairs of neighboring vertices with either equal abscissas or equal ordinates, where two vertices with equal abscissas (ordinates) are neighboring if they are not separated by another vertex with the same abscissa (ordinate). For the moment, denote the vertices of  $L$  by symbols  $a, b, c, \dots, f$  and the arcs by symbols such as  $ab, bc, \dots, cf$  (no distinction is made between the arcs  $ab$  and  $ba$ ). Then a chain is a set of one or more distinct arcs that can be arranged as  $ab, bc, \dots, de, ef$ , where vertices denoted by different symbols are distinct. A cycle is a set of distinct arcs (at least four are necessary) that can be ordered as  $ab, bc, \dots, ef, fa$ , the vertices being distinct as in the case of a chain. A graph is connected if each pair of vertices is joined by a chain. A forest is a graph containing no cycles, and a tree is a connected forest.

If  $L$  contains  $v$  vertices,  $a$  arcs, and  $p$  connected pieces, the number  $\mu = a - v + p$  is known as the cyclomatic number (or first

Betti number) of  $L$ . It follows from a well known theorem [8] concerning linear graphs in general that: (i)  $L$  is a forest if and only if  $\mu = 0$ , and (ii)  $L$  contains just one cycle if and only if  $\mu = 1$ .

Note that  $L$  contains a cycle if and only if there is a subset of  $I$  that can be arranged as a sequence

$$s_{h_1 k_1}, s_{h_1 k_2}, s_{h_2 k_2}, s_{h_2 k_3}, \dots, s_{h_\sigma k_\sigma}, s_{h_\sigma k_1}$$

where the  $h$ 's and  $k$ 's are distinct among themselves; and  $L$  contains a single cycle if and only if  $I$  contains just one subset that can be arranged in the displayed form. We call such a subset of  $I$  an I-circuit on  $S$ , and denote it by  $[S_\sigma]$ . For a particular arrangement of  $[S_\sigma]$ , we shall also refer to the terms  $s_{h_\alpha k_\alpha}$  as odd-terms, the others as even-terms.

In case  $I$  consists of all  $s_{hk} > 0$ , as it frequently will, we speak of the positive graph of  $S$ , positive circuits on  $S$ , and abbreviate such statements as "the positive graph of  $S$  is a forest" to " $S$  is a forest".

## 6. The Method of Solution

In the method of solution to be developed for the problem, we start with a special set of values  $X^0 = \{x_{ij}^0\}$  that satisfy the constraints (4.6). We then test to determine whether or not there exist  $u_i$  and  $v_j$  satisfying the relations (4.7) and (4.8) for the given  $X^0$ . If so, then  $X^0$  is a solution, otherwise not. The method next yields a new trial matrix  $X^1 = \{x_{ij}^1\}$ , if  $X^0$  is not a solution, such that  $\sum_{i,j} (x_{ij}^0 - x_{ij}^1) d_{ij} \geq \dots$ . After a finite number of steps this



process necessarily must terminate and it leads to an exact integral solution of the problem.

The first trial matrix  $X^0$  is a forest of  $t$  trees, and has  $m+n-t$  non-zero elements. According as  $t = 1$  or  $t > 1$ , two essentially different cases may be met at each stage of the solution process.<sup>6</sup>

At each stage when  $X = \|x_{ij}\|$  is a tree, the equations (4.8) have a general solution for  $u_i$  and  $v_j$  with one free parameter, say  $u_1$ . However, the quantities  $d_{ij} + u_i - v_j$  are uniquely determined in this case, so it is sufficient to calculate them and note whether or not they are all non-negative in order to decide whether or not  $X$  is a solution. If some  $d_{i_1 j_1} + u_{i_1} - v_{j_1} < 0$ , then there is a unique I-circuit  $[X_g]$  on  $X$ , where  $I$  consists of  $x_{i_1 j_1}$  and all positive  $x_{ij}$ , that may be arranged with  $x_{i_1 j_1}$  as the second term, say. Let  $g$  denote the smallest odd-term of  $[X_g]$ . Then the new trial matrix  $X^*$  is obtained from  $X$  by adding  $g$  to the even terms of  $[X_g]$ , subtracting  $g$  from the odd-terms, and leaving the other elements of  $X$  unchanged.

At each stage when  $X$  is a forest of  $t > 1$  trees, the equations (4.8) have a general solution for  $u_i$  and  $v_j$  with  $t$  independent parameters, and the quantities  $d_{ij} + u_i - v_j$  involve  $t-1$  independent parameters. The rows and columns of the matrix  $X$  are rearranged so that it can be represented as a square matrix of order  $t$  whose  $t^2$  elements are submatrices  $X_{ab}$  such that  $X_{ab} = 0$  if  $a \neq b$  and  $X_{aa}$  is a tree with  $m_a + n_a - 1$  non-zero elements and is of order  $m_a \times n_a$ . We can select  $u_1, u_{m_1+1}, \dots, u_{m_1+\dots+m_{t-1}+1}$  to be the  $t$  parameters. If we assign these the value zero and denote this particular

solution of (4.8) by  $\bar{u}_i$  and  $\bar{v}_j$ , then we may define numbers  $\bar{p}_{ij} = d_{ij} + \bar{u}_i - \bar{v}_j$ . We partition the matrix  $\bar{P} = \|\bar{p}_{ij}\|$  into submatrices corresponding to the  $X_{ab}$  and denote them  $\bar{P}_{ab}$ . Let  $p_{ab}$  be the smallest element in  $\bar{P}_{ab}$  and define the square matrix  $P$  of order  $t$  by  $P = \|p_{ab}\|$ . To designate the position of  $p_{ab}$  in the matrix  $\bar{P} = \|\bar{p}_{ij}\|$ , we may write  $p_{ab}$  alternatively as  $\bar{p}_{a^i b^j}$ , the subscripts referring to the submatrix and the superscripts to the rows and columns in the submatrix.

The test as to whether or not  $X$  is a solution consists of forming all sums  $p_{a_1 a_2 \dots a_n} = p_{a_1 a_2} + p_{a_2 a_3} + \dots + p_{a_n a_1}$  for  $n = 2, 3, \dots, t$ , where  $(a_1 a_2 \dots a_n)$  is any permutation of  $n$  different positive integers, none greater than  $t$ ;  $X$  is a solution if and only if all such sums are non-negative.

If any  $p_{a_1 a_2 \dots a_n} < 0$ , then there is a unique I-circuit  $[X_g]$  on  $X$ , where  $I$  consists of all positive  $x_{ij}$  together with all  $x_{ij}$

that correspond to the terms  $\bar{p}_{a_k^i a_{k+1}^j}$  of  $p_{a_1 a_2 \dots a_n}$ ,

which can be arranged to involve all  $x_{a_k^i a_{k+1}^j}$  as

even-terms. If  $g$  is the smallest odd-term in  $[X_g]$ , then (as in the non-degenerate case) the new trial matrix  $X^*$  is obtained by adding  $g$  to the even-terms of  $[X_g]$ , subtracting  $g$  from the odd-terms, and leaving the other elements of  $X$  unchanged.

## 7. The Initial Trial Solution

An  $X$  that satisfies (4.6) will be called a trial solution. It would be all right to take the positive values  $r_i c_i / \sum r_i$  for

the initial trial solution  $X^0 = \|x_{ij}^0\|$ . An alternative is to construct an initial trial solution that is a forest. It is always possible to do this in integral values. The following theorem certifies the existence of such an integral trial solution. The method of proof shows how to construct one.

Theorem 3. There is a matrix  $X^0 = \|x_{ij}^0\|$  with integral elements that satisfies (4.6) and is a forest.

Proof: The theorem is trivial for  $m=1$ . Assume the theorem is true for  $m$  and consider the case  $m+1$ .

Let the notation be chosen so that  $r_1 \geq r_2 \geq \dots \geq r_{m+1} > 0$ , and  $c_1 \geq c_2 \geq \dots \geq c_n > 0$ . If  $n < m+1$ , then  $c_1 > r_{m+1}$ . If  $n = m+1$  then  $c_1 > r_{m+1}$  unless  $c_1 = r_j = \lambda$  (for all  $i$  and  $j$ ); in this latter case  $X^0 = \lambda$  satisfies the conditions of the theorem. Hence, by the induction hypothesis, there is a set of non-negative integers  $x_{ij}^*$  ( $i = 1, \dots, m$ ) such that  $\sum_j x_{ij}^* = c_j - \delta_{1,j} r_{m+1}$ ,  $\sum_i x_{ij}^* = r_i$ , and  $X^* = \|x_{ij}^*\|$  is a forest. Then  $X^0$  defined by  $x_{ij}^0 = x_{ij}^*$ ,  $x_{m+1,j}^0 = \delta_{1,j} r_{m+1}$ , satisfies (4.6). Now since the  $(m+1)^{st}$  row, with only one positive element, clearly cannot contribute terms to a positive circuit,  $X^0$  is also a forest; and the theorem is proved.

To apply this method, in the construction of a trial solution, search for the smallest  $r_{i_1}$  and the largest  $c_{j_1}$ , and then set  $x_{i_1 j_1}^0 = r_{i_1}$ . In effect, this deletes the  $i_1^{st}$  row, after  $c_{j_1}$  is replaced by  $c_{j_1} - r_{i_1}$ , and the process is repeated (interchanging rows and columns as necessary) until all  $x_{ij}^0$  have been determined. For automatic machine calculation the procedure is easily made unique, for any one starting order of rows and columns, by

specifying that the search is first on row-totals when the number of rows is the same as the number of columns at any stage, and that the row-total or column-total with the smallest index is chosen whenever at any stage there are several equal values to choose from. This initial trial solution will be called "preferred" for identification.<sup>7</sup>

Theorem 4. A trial solution that is a forest of  $t$  trees has  $m+n-t$  non-zero elements.

Proof: Observe first that if the trial solution  $X$  is a forest of  $t$  trees, the rows and columns of  $X$  can be rearranged so that  $X$  has the form

$$\left\| \begin{array}{cccc} X_{11} & 0 & \cdots & 0 \\ 0 & X_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & X_{tt} \end{array} \right\|,$$

where each  $X_{aa}$  is a tree. Consequently, the theorem amounts to proving that an  $m \times n$  matrix with no zero rows or columns, which is a tree, has  $m+n-1$  positive elements. If  $m+n = 2$ , this is obvious, so assume the statement to be true for all matrices for which  $m+n = k$  and consider one for which  $m+n = k+1$ . Since  $m \geq n$ , clearly some row has only one positive element, as otherwise there would be a positive circuit. Delete this row and apply the induction hypothesis.

In actual cases when  $m$  and  $n$  are relatively small, or when there is other reason to believe that an initial trial solution better than the preferred one can be found by trial and error, it may be better to construct the initial trial solution in some other

way than the one given in the proof of Theorem 3, in order to reduce the number of steps required in the iterative process.

The methods developed in this paper apply directly for any trial solution that is a forest, and are readily extended for other cases. It is easy to see that there must be at least one solution which is a forest.

### 8. Non-degenerate Case

We consider now the case of a trial solution  $X$  which is a tree. Let the positive elements of  $X$  be  $x_{i_a j_a}$ ,  $a = 1, \dots, m+n-1$ . We shall need the following theorem.

Theorem 5. If  $X$  is a trial tree, the set of equations

$$(8.1) \quad d_{ij} + u_i - v_j = 0 \quad \text{for } (i, j) = (i_a, j_a),$$

has the general solution

$$u_i = u_i^* + z, \quad v_j = v_j^* + z,$$

where  $(u_i^*, v_j^*)$  is a particular solution and  $z$  is arbitrary.

Proof: The theorem is apparent for  $m = 1$ , and we proceed by induction. Suppose the theorem is true for all trial trees of  $m$  rows, and let  $X$  be an  $(m+1) \times n$  trial tree. Obviously, there must be at least one row of  $X$  that has exactly one non-zero element; we may suppose it to be  $x_{m+1, n}$  without loss of generality—also that  $i_{m+n-1} = m+1$  and  $j_{m+n-1} = n$ . Since  $X$  is a trial tree, the matrix obtained from  $X$  by deleting the last row (or, if  $m+1 = n$ , its transpose) is also. The induction hypothesis implies that the general solution of (8.1), with the final equation omitted, is of the form  $u_i = u_i^* + z$ ,  $v_j = v_j^* + z$ . We note next that this final equation becomes  $u_{m+1} = (v_n^* - d_{m+1, n}) + z = u_{m+1}^* + z$ . The theorem follows easily.

It will be convenient to call the particular solution  $\bar{u}_i, \bar{v}_j$  of (8.1) obtained by setting  $u_i = 0$  the preferred trial solution

of the dual problem corresponding to the trial tree  $X$ . As an obvious consequence of Theorem 5, we state

Corollary 5A. If  $X$  is a trial tree, then it is a solution of the problem if and only if the corresponding preferred trial solution  $(\bar{u}_1, \bar{v}_j)$  of the dual problem satisfies  $d_{1j} + \bar{u}_1 - \bar{v}_j \geq 0$  (for all  $i$  and  $j$ ).

All that is needed now in order to establish the method for the non-degenerate case is to show how to construct a new trial matrix  $X^*$ , if  $X$  is not a solution, such that  $\sum_{i,j} (x_{1j} - x_{1j}^*) d_{1j} \geq 1$ . In this case, it follows by Corollary 5A that  $d_{kl} + \bar{u}_k - \bar{v}_l < 0$  for at least one pair  $(k, l)$  and, of course,  $x_{kl} = 0$ .

Theorem 6. If the trial solution  $X$  is a tree, and  $x_{kl} = 0$ , then there is a unique I-circuit on  $X$ , where  $I$  consists of all positive  $x_{1j}$  together with  $x_{kl}$ .

Proof: It suffices to show that the I-graph of  $X$  has cyclomatic number  $\mu = 1$ . By assumption, the positive graph of  $X$  has cyclomatic number zero, and since  $X$  must have positive elements  $x_{al}$  and  $x_{kb}$  for some  $a$  and  $b$ , the I-graph of  $X$  has two more arcs, one more vertex, and the same number (one) of connected pieces. Hence  $\mu = 1$ , and the proof is complete.

Now arrange this unique I-circuit  $[X_g]$  with  $x_{kl}$  as the second term and let  $g$  be the minimum of the odd-terms of  $[X_g]$  in this arrangement. If we subtract  $g$  from the odd-terms, add  $g$  to the even terms, and leave the remaining elements of  $X$  unchanged, we get a matrix  $X^*$  that satisfies (4.5) and is a forest (since  $[X_g]$  was unique).

Theorem 7.  $\sum_{1,j} (x_{1j} - x_{1j}^*) d_{1j} \geq 1.$

Proof: Let  $[X_s] = [x_{1,j_1}, x_{1,j_2}, x_{1,j_2}, x_{1,j_3}, \dots, x_{1,j_s}, x_{1,j_1}]$ ,

where  $x_{1,j_2} = x_{k,l}$ . Then

$$\begin{aligned} \sum_{1,j} (x_{1j} - x_{1j}^*) d_{1j} &= g(d_{1,j_1} - d_{1,j_2} + d_{1,j_2} - d_{1,j_3} + \dots + d_{1,j_s} - d_{1,j_1}) \\ &= -g(d_{1,j_2} + u_{1,j_1} - v_{1,j_s}) \geq 1. \end{aligned}$$

The theorem follows. |

If  $X^*$  is a tree, then the whole process is repeated until at some stage a trial matrix is obtained that either: (i) is a solution, or (ii) is not a solution and is a forest of  $t > 1$  trees. We discuss (ii) next.

### 9. The Degenerate Case

Let  $X$  be a trial matrix which is a forest of  $t > 1$  trees. As we have seen, we may suppose that the rows and columns of  $X$  are ordered so that

$$X = \left\| \begin{array}{cccc} X_{11} & 0 & \dots & 0 \\ 0 & X_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X_{tt} \end{array} \right\|,$$

where each submatrix  $X_{aa}$  of order  $m_a \times n_a$  is a tree. We can apply the methods of the non-degenerate case to the subproblems corresponding to the submatrices  $X_{aa}$  and either obtain a solution to each subproblem or further decompose the matrix  $X$ , so we may also assume that each  $X_{aa}$  is a solution to its subproblem.

By Corollary 5A, we know that

$$(9.1) \quad d_{a a}^{i_a j_a} + \bar{u}_a^{i_a} - \bar{v}_a^{j_a} \geq 0, \quad a = 1, \dots, t; \quad i_a = 1, \dots, m_a; \quad j_a = 1, \dots, n_a,$$

where  $\bar{u}_a, \bar{v}_a$  is the preferred trial solution of the dual subproblem corresponding to the solution  $X_{aa}$ , and that

$$(9.2) \quad d_{a a}^{i_a j_a} + \bar{u}_a^{i_a} - \bar{v}_a^{j_a} = 0 \quad \text{if} \quad x_{a a}^{i_a j_a} > 0.$$

We recall also that the most general values for  $u_a^{i_a}$  and  $v_a^{j_a}$  are given by

$$u_a^{i_a} = \bar{u}_a^{i_a} + z_a, \quad v_a^{j_a} = \bar{v}_a^{j_a} + z_a,$$

where the  $z_a$  are arbitrary parameters.

It follows from Theorem 2 that  $X$  is a solution if and only if there are values of  $z_a$  that satisfy inequalities corresponding to (4.7), or in our present notation:

$$(9.3) \quad d_{a b}^{i_a j_b} + u_a^{i_a} - v_b^{j_b} \geq 0 \quad \text{for all } a, b, i_a, \text{ and } j_b.$$

But (9.3) has a solution for  $z_a$  if and only if the following inequalities have a solution for  $z_a$ :

$$(9.4) \quad p_{ab} + z_a - z_b \geq 0,$$

where

$$\bar{p}_{a b}^{i_a j_b} = d_{a b}^{i_a j_b} + \bar{u}_a^{i_a} - \bar{v}_b^{j_b},$$

$$p_{ab} = \min_{(i_a, j_b)} \left\{ \bar{p}_{a b}^{i_a j_b} \right\}.$$

We have proved

Lemma A.  $X$  is a solution if and only if there are real numbers  $z_a$  such that  $p_{ab} + z_a - z_b \geq 0$ ,  $a, b = 1, \dots, t$ .



In order to establish a criterion for the solvability of (9.4), we consider a special case of the original problem, defined as follows:  $d_{ab} = p_{ab}$ ,  $r_a = c_b = 1$ ,  $a, b = 1, \dots, t$ . We call this the special problem, the corresponding dual the special dual, and now consider the special combined problem:

$$\sum_b y_{ab} = \sum_a y_{ab} = 1, \quad y_{ab} \geq 0,$$

$$p_{ab} + z_a - w_b \geq 0,$$

$$y_{ab}(p_{ab} + z_a - w_b) = 0.$$

If we set  $y_{ab} = \delta_{ab}$ , then for this trial solution the conditions reduce to:

$$p_{ab} + z_a - w_b \geq 0 \quad \text{for } a \neq b,$$

$$p_{aa} + z_a - w_a = 0.$$

Since  $p_{aa} = 0$ , it follows that  $z_a = w_a$ , and so these conditions are equivalent to (9.4). Hence, by Theorem 2, (9.4) has a solution if and only if  $\|\delta_{ab}\|$  is a solution of the special problem. Using Lemma A, we now have

Lemma B.  $X$  is a solution of the original problem if and only if the identity matrix is a solution of the special problem.

Theorem 8.  $X$  is a solution of the problem if and only if

$p_{a_1 a_2 \dots a_h} \geq 0$ ,  $h = 2, 3, \dots, t$ , where  $(a_1, a_2, \dots, a_h)$  is any permutation of  $h$  different positive integers, none greater than  $t$ , and

$$p_{a_1 a_2 \dots a_h} = p_{a_1 a_2} + p_{a_2 a_3} + \dots + p_{a_h a_1}.$$

Proof: By Lemma B, it suffices to show that the condition of the theorem is equivalent to the statement that  $\|\delta_{ab}\|$  is a solution of the special problem.

First of all, it is easy to see that at least one solution  $Y = \|y_{ab}\|$  of the special problem is a forest, and hence has less than  $2t$  non-zero elements. That the elements of  $Y$  are all either

zero or unity can be seen by induction as follows. The basis of the induction is obvious, and we consider the case  $t+1$ , assuming the statement for  $t$ . There must be at least one element of  $Y$  that is unity, as otherwise  $Y$  would have at least  $2t$  non-zero elements. We may suppose that this element is  $y_{t+1, t+1}$ . But then the induction hypothesis implies that each element  $y_{ab}$ ,  $a, b = 1, \dots, t$ , is zero or one. It follows that there are exactly  $t$  elements of  $Y$  that are unity, whence  $\sum_{a,b} y_{ab} p_{ab}$  can be written as  $p_{a_1 b_1} + p_{a_2 b_2} + \dots + p_{a_t b_t}$  where  $(a_1 a_2 \dots a_t)$  and  $(b_1 b_2 \dots b_t)$  are permutations of the first  $t$  integers. Then  $\|\delta_{ab}\|$  is a solution of the special problem if and only if always

$$p_{a_1 b_1} + p_{a_2 b_2} + \dots + p_{a_t b_t} \geq p_{11} + p_{22} + \dots + p_{tt} = 0.$$

The proof is completed by noting that this sum can be written as  $p_{a_1 a_2} + p_{a_2 a_3} + \dots + p_{a_h a_1}$ , with  $(a_1 a_2 \dots a_h)$  as described in the theorem.

We now need to show how to construct an improved trial solution  $X^*$  in the event that  $X$  is not a solution. In this case, we know from Theorem 8 that there is a sum

$$\frac{1^0_{a_1} j^0_{a_2}}{p_{a_1 a_2}} + \frac{1^0_{a_2} j^0_{a_3}}{p_{a_2 a_3}} + \dots + \frac{1^0_{a_h} j^0_{a_1}}{p_{a_h a_1}} < 0.$$

Let  $I$  consist of all positive elements  $x_{a_a}^{1^0 j^0_a}$  together with all  $x_{a_k a_{k+1}}^{1^0 j^0_{a_{k+1}}}$  of  $X$ . Then we assert:

Theorem 9. There is a unique  $I$ -circuit on  $X$  that can be

arranged to involve all the  $x_{a_k a_{k+1}}^{1^0 j^0_{a_{k+1}}}$  as even-terms.

Proof: The positive graph of  $X$  has  $m+n-t$  vertices,  $m+n-2t$  arcs, and  $t$  connected pieces. Also, for each  $x_{a_k a_{k+1}}^{i_a^0 j_{a_{k+1}}^0}$  there are non-zero elements  $x_{a_k a_k}^{i_a^1 j_{a_k}^1}$ ,  $x_{a_{k+1} a_{k+1}}^{i_{a_{k+1}}^1 j_{a_{k+1}}^0}$ . Hence in passing from the positive graph to the I-graph,  $h$  vertices and  $2h$  arcs are added, and the number of connected pieces is decreased from  $t$  to  $t-h+1$ . Thus the cyclic number of the I-graph is

$$\mu = (2h+m+n-2t) - (h+m+n-t) + (t-h+1) = 1,$$

so there is a unique I-circuit  $[X_g]$  on  $X$ . Since the graph obtained by omitting from  $I$  any  $x_{a_k a_{k+1}}^{i_a^0 j_{a_{k+1}}^0}$  clearly has no cycle,  $[X_g]$  contains all of these.

Evidently  $[X_g]$  can be arranged, for example, as

$$[x_{a_1 a_1}^{i_{a_1}^0 j_{a_1}^0}, x_{a_1 a_2}^{i_{a_1}^0 j_{a_2}^0}, x_{a_2 a_2}^{i_{a_2}^0 j_{a_2}^0}, \dots, x_{a_2 a_2}^{i_{a_2}^0 j_{a_2}^0}, x_{a_2 a_3}^{i_{a_2}^0 j_{a_3}^0}, \dots]$$

so that all  $x_{a_k a_{k+1}}^{i_a^0 j_{a_{k+1}}^0}$  appear as even-terms.

As in the non-degenerate case, let  $g$  be the smallest odd-term in  $[X_g]$  (hence  $g > 0$ ) and define a new trial matrix  $X^*$  by replacing the elements of  $X$  that appear in  $[X_g]$  by new ones increased by  $g$  for even-terms and decreased by  $g$  for odd terms; the other elements of  $X$  are left unchanged. Again  $X^*$  satisfies the conditions for a trial matrix. To complete the discussion of the degenerate case, it remains only to prove

Theorem 10.  $\sum_{i,j} (x_{ij} - x_{ij}^*) d_{ij} \geq 1.$

Proof: Since  $X$  and  $X^*$  differ only on

$$[X_s] = [x_{1,j_1}, x_{1,j_2}, x_{1,j_3}, \dots, x_{1,j_s}, x_{1,j_1}],$$

then

$$\sum_{1,j} (x_{1,j} - x_{1,j}^*) d_{1,j} = -g(d_{1,j_1} - d_{1,j_2} + d_{1,j_2} - d_{1,j_3} + \dots + d_{1,j_s} - d_{1,j_1}).$$

The proof is completed by noting that  $d_{1,j} = \bar{p}_{1,j} + \bar{v}_j - \bar{u}_1$  and

$$\bar{p}_{1,j} = 0 \text{ if } x_{1,j} > 0, \text{ so that } \sum_{1,j} (x_{1,j} - x_{1,j}^*) d_{1,j} = -g(p_{a_1, a_2, \dots, a_n}) \geq 1.$$

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# Footnotes

1. Numbers in brackets refer to bibliography at the end of the paper.
2. My interest in the problem was aroused by papers on transportation theory presented by Koopmans [4a] and Dantzig [4b] at a conference on linear programming in Chicago during June 1949, under the auspices of the Cowles Commission for Research in Economics of the University of Chicago. Several other papers presented at this conference are of closely related interest. Professor Koopmans, in his introduction to the Conference Proceedings [4], also discusses the background and interrelationship of the conference papers—including the bearing of some of these on the Hitchcock distribution problem. The results of the present paper have been presented in three seminar lectures: once in December 1949 at The RAND Corporation in Santa Monica, once in July 1950 at the Institute for Numerical Analysis of the National Bureau of Standards in Los Angeles, and once in June 1951 at the National Bureau of Standards in Washington, D. C.

I am especially indebted to Dr. D. R. Fulkerson, who has given real assistance in simplifying notation and proofs of theorems, for a careful reading of the manuscript.

3. There is no loss of generality in assuming that the  $d_{ij}$  are positive integers, rather than rational numbers, since the problem is essentially unchanged if  $d_{ij}$  is replaced by  $ad_{ij} + b$  where  $a$  and  $b$  are any positive rational numbers. I have not examined the case in which some of the quantities  $r_i$ ,  $c_j$ , and  $d_{ij}$  are irrational. The only effect of irrationality on the results of the present paper is a possible lack of convergence of the iterative process of solution. These considerations are not of importance in the usual applications.

4. Unpublished note, December 1949.

5. We omit the condition (4.2), that  $x \geq 0$ , since this imposes no limitation on  $u_i$  and  $v_j$ .

6. These are the non-degenerate and degenerate cases in the work of Dantzig [4b]. I shall use these terms also. The method of solution developed by Dantzig [4b] for the non-degenerate case is essentially the same as the one in this paper, although the derivations of the results are quite different. Orden [7] has subsequently given an elegant method for reducing the degenerate case to the non-degenerate one, as an extension of the  $\epsilon$ -method proposed by Dantzig [4b]. I believe that the treatment of the degenerate case provides the only results in the present paper that are new, or at least fresh for the Hitchcock problem, and also of some mathematical interest. It also seems likely that the method given here will often be more efficient computationally, in the degenerate case, than the Dantzig-Orden  $\epsilon$ -method.

7. Sometimes, as in this instance, I indicate how to make a unique choice among possible alternatives at each computational step but usually I do not. It is necessary to do this in order completely to routinize the computing steps, of course, but the matter presents no difficulty and I omit it here.

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